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Mathematics

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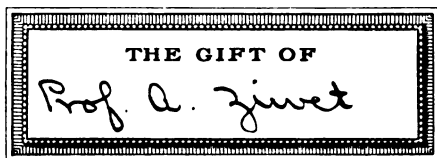
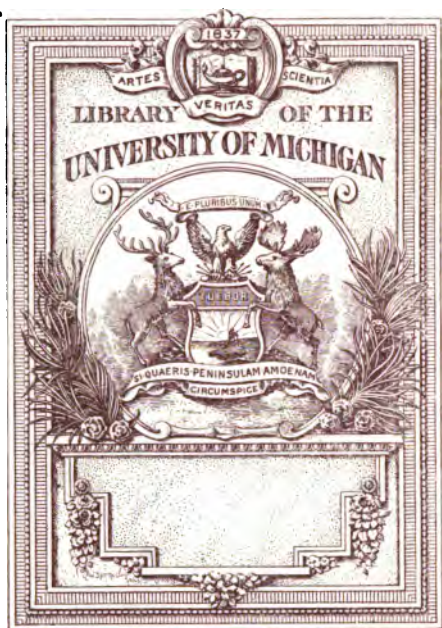
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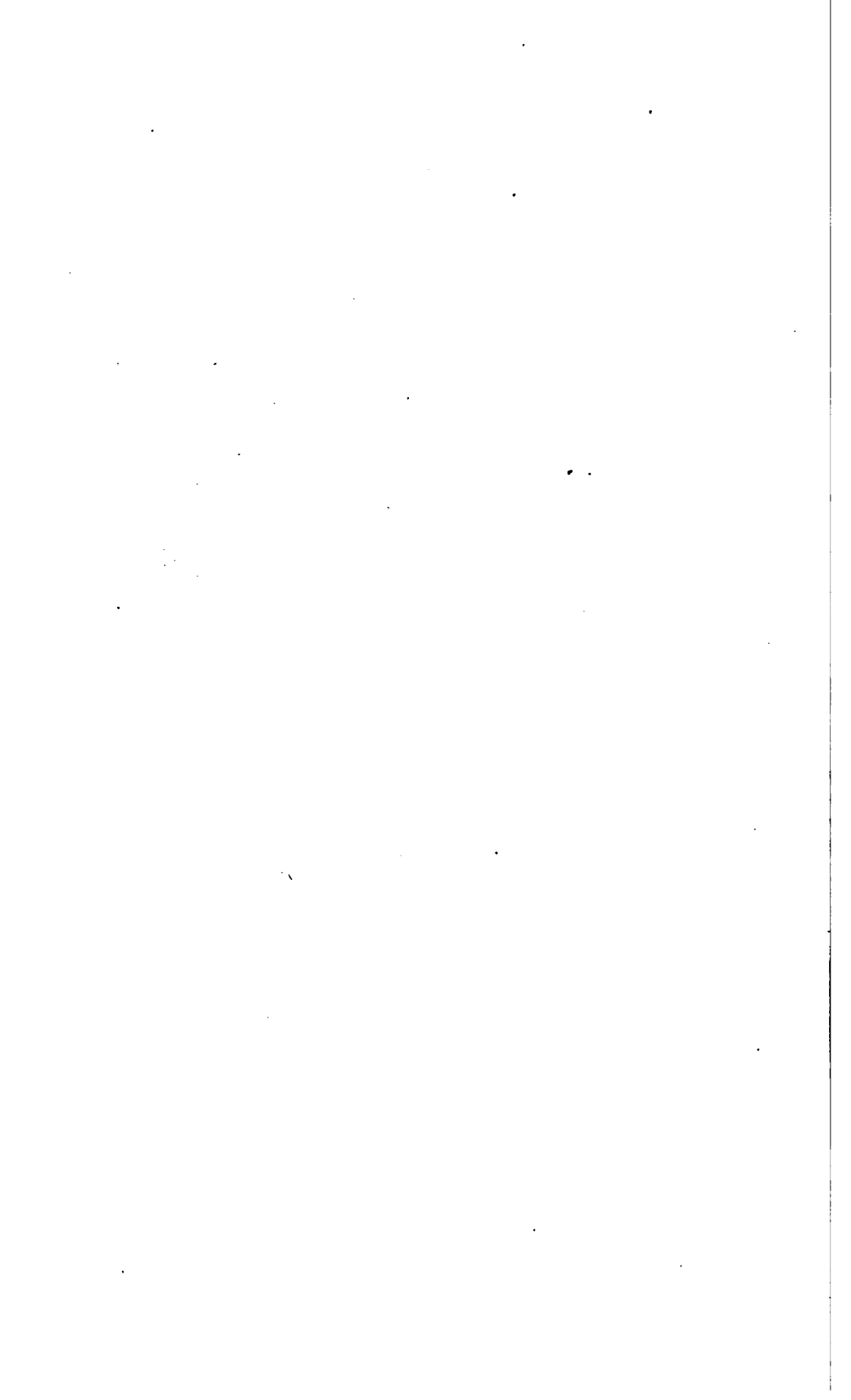
Hedrick - Pamphlets -



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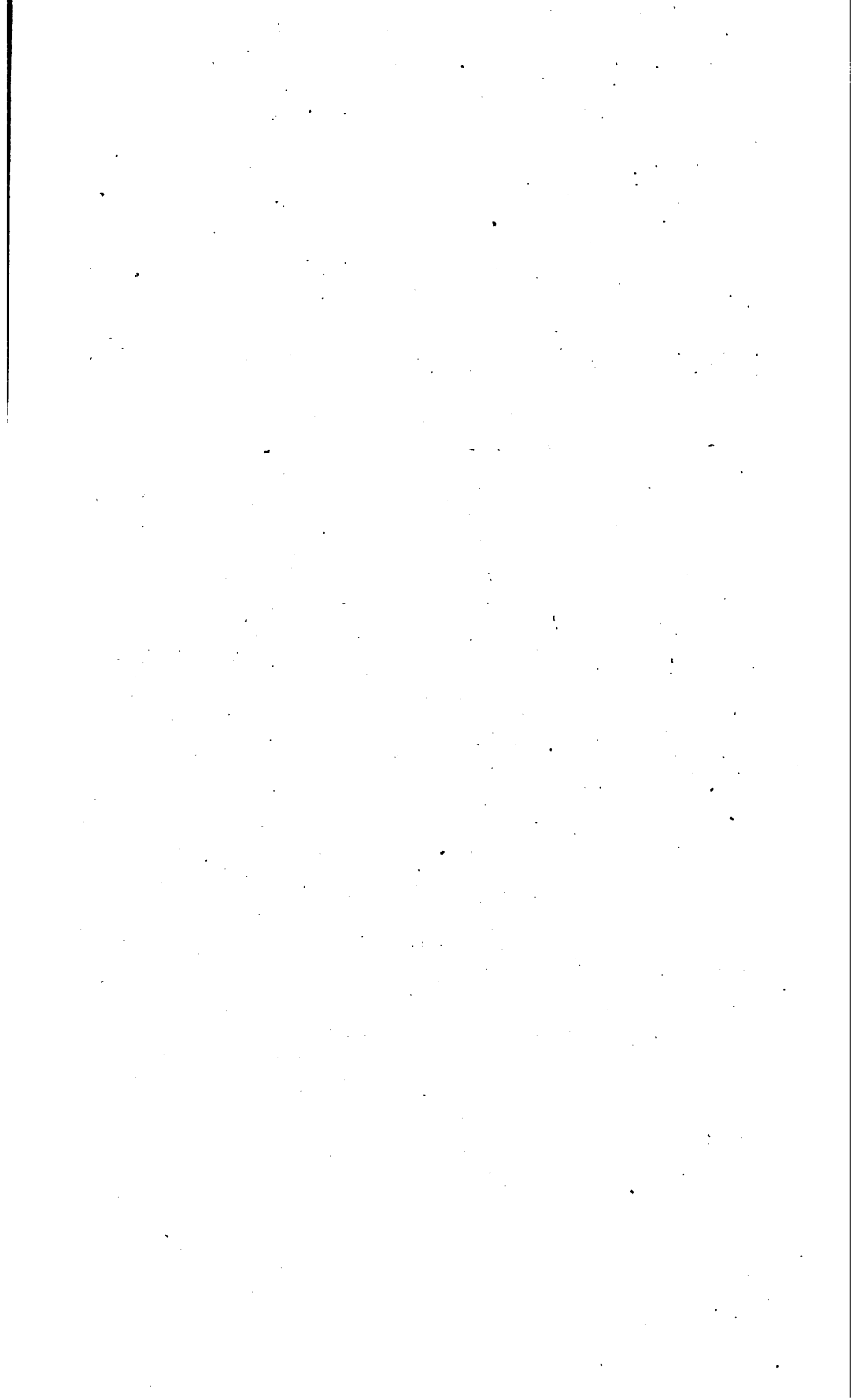


Alexander Ziwet

THE ENGLISH AND FRENCH TRANSLATIONS OF
HILBERT'S GRUNDLAGEN DER GEOMETRIE.

BY DR. E. R. HEDRICK.

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Sept, 13 1906

THE ENGLISH AND FRENCH TRANSLATIONS OF
HILBERT'S GRUNDLAGEN DER GEOMETRIE.

1. *Les Principes fondamentaux de la Géométrie.* Par D. HILBERT. Traduit par L. LAUGEL. Paris, Gauthier-Villars, 1900. 4to. 111 pp.
2. *The Foundations of Geometry.* By DAVID HILBERT. Translated by E. J. TOWNSEND. Chicago, The Open Court Publishing Company, 1902. 8vo. 132 pp.

It is indeed a matter for congratulation that Professor Hilbert's masterly discussion of the foundations of geometry has become so well known and so widely circulated. This circumstance is undoubtedly due to its clearness and force; for, while some of the previous studies along similar lines are difficult even for the advanced student to understand, Hilbert's style is so deceptively clear as to lead certain minds to predict the use of this book in elementary instruction.

An excellent review of the original,* rendered into English by Professor Ziwet, was given in the BULLETIN (volume 6, 1900, pages 287-299) by Dr. Sommer, of Göttingen; the present review will therefore not deal with the German edition, except for purposes of comparison.

If a minute criticism of the language of the French translation were the main purpose, the reviewer would certainly feel great hesitation in undertaking the task. But there are certain additions to the French translation which are most noteworthy, and with these we shall occupy ourselves chiefly.

On page 291 of his review, mentioned above, Dr. Sommer explains that Hilbert's axiom V is not sufficient to furnish a satisfactory foundation for the complete discussion of "the continuity of the straight line in the ordinary sense." This lack is supplied in the French edition by an additional axiom (by Hilbert) entitled "Axiome d'intégrité" (Vollständigkeitsaxiom), which practically requires that the system already set up shall

* In this connection, a review of Hilbert's *Grundlagen der Geometrie* by H. Poincaré, *Bull. des Sciences Math.* (2), vol. 26 (Sept., 1902), p. 249, should be mentioned. This review is, as would be expected, of the greatest importance. The present writer regrets that his review was in type before that of Poincaré was seen. Footnotes have been added occasionally at points where this review would have been most influenced by Poincaré's.

be considered as closed, *i. e.*, incapable of extension by the adjunction of additional elements. This axiom corresponds to the one introduced by Hilbert (*Jahresbericht der Deutschen Mathematiker Vereinigung*, 1900) for arithmetic; and is certainly radically different, in the character of the assumption made, from the other axioms proposed by Hilbert. Its introduction enables us to prove the fundamental theorem of Bolzano, but the reviewer feels that its form is scarcely final, and that it is not wholly satisfactory to Hilbert himself.*

Another addition to the French edition merits more attention than it seems to have received; the original conclusion of the German edition is replaced by a new appendix (pages 106–111), written by Hilbert, which shows clearly the importance of his work for non-euclidean geometry in its broadest sense, a point upon which Hilbert dwelt to a considerable extent in his lectures preceding the publication of the original *Festschrift*, but which is scarcely mentioned in the *Festschrift* itself. The apparent reason for the tendency of non-euclidean geometers in the past to cast out the axiom of parallels alone, is that this was thought to be the only axiom *explicitly* assumed by Euclid which could not be directly tested by physical experiment.† Another axiom, or set of axioms, namely those regarding continuity, assumed *implicitly* by Euclid, is, however, equally open to philosophical exception.‡ The effect of the omission of the axiom of Archimedes (Hilbert, V) was therefore of peculiar interest to Hilbert, and the subject was discussed considerably in his lectures. A student, Dr. Dehn, completed these inquiries, and a review (by Hilbert) of Dr. Dehn's thesis forms the major portion of this addition to the French translation. The one striking fact that *a geometry is possible in which the sum of the angles of any triangle is two right angles and in which similar non-congruent triangles exist, while an infinity of parallels to any straight line may be drawn through any point*, will sufficiently indicate the remarkable nature of Dr. Dehn's results.

This geometry is called "semi-euclidean," and is evidently non-archimedean, the axioms III and V being omitted; but in the whole discussion the axioms I, IV and II (in modified form) are assumed to hold. To the reviewer it would seem, from the theorem just mentioned, that the assumption that the

* See also Poincaré, l. c., pp. 256, 271, 272.

† The reviewer disclaims any intention of saying that this is a logically valid reason; it is merely the *actual* reason.

‡ See Poincaré, l. c., pp. 250, 258.

sum of the angles of a triangle is two right angles would be more advisable than the form of the parallel axiom given by Euclid (Hilbert III), since it is demonstrated that it is a "weaker" axiom, in the sense of Professor Moore. For while it can be proved from the axiom III, without the use of axiom V, we must include V in order to prove III from it. At any rate, as Dr. Dehn remarks, this substitute for the axiom of parallels is conclusively shown not to be equivalent to Euclid's form; and, while Euclid probably deserves no praise in the matter, his was certainly a happy choice in that he possessed Archimedes's axiom only by implicit assumption. While this subject merits considerable discussion, we cannot enter into further details without passing from the book in hand. Still, Dr. Dehn's thesis (*Mathematische Annalen*, volume 53 (1900)), as also his subsequent papers in the *Annalen*, form a direct and important addition to Hilbert's memoir.

Aside from these additions, the French edition is simply a remarkably good translation. The translator has happily given himself just enough liberty to render the spirit of the German into idiomatic French; but the reviewer feels confident that the sense has always been perfectly preserved. It is especially noteworthy that M. Laugel had the direct and enthusiastic assistance of Hilbert himself in the preparation of the French edition.

In spite of the hesitation expressed above, the reviewer feels moved to enquire whether better translations for "Axiome der Verknüpfung" (axiomes d'association), and for "Axiome der Anordnung" (axiomes de distribution), could not have been found, especially in view of the necessity of using these same terms in describing the corresponding sets of axioms of arithmetic in Chapter III, where the associative, distributive, and commutative laws of arithmetic are all placed under the head of "associative" axioms, whereas the "distributive" axioms are of an entirely different nature.* Finally the translations of "flächengleich" (égaux par addition), and of "inhaltsgleich" (égaux par soustraction), are certainly widely divergent from the original, and might possibly have been improved upon; but it may be contended that they are justifiable as a closer description of the ideas represented.

Of the typography nothing need be said, further than that the book is printed by the firm of Gauthier-Villars.

* Poincaré, l. c., uses "axiomes projectifs" and "axiomes de l'ordre." He expressly discards "axiomes de connection" as a translation of "Axiome der Verknüpfung."

On the whole, on account of the additions mentioned above, and in view of the faithful reproduction of the spirit of the original, the French edition compares very favorably with the German, for those who understand both languages equally well. The reviewer had excellent success in the use of the French edition by a portion of his class last winter, the students having been advised to procure either the French or the German editions at their discretion.

The English translation is the work of one who had the advantage of hearing Hilbert's preparatory lectures. It contains the additions to the French translation, but there are no new additions. Some unimportant omissions are made, including the original conclusion (also omitted in the French edition). In a short preface Professor Townsend expresses his own views upon the important features of the work. It is unfortunate that he does not emphasize the radical departures of Hilbert's *method* of presentation — especially the power obtained by considering the axioms as fundamental, the conceptions of point, line, and plane as trivial.* Notice is taken in footnotes of some recent memoirs commenting on Hilbert's work.

Some features of the translation are particularly happy. Especially does the rendering of "Widerspruchslosigkeit" by "compatibility" appeal to the reviewer. The translations of "flächengleich" (of equal area), and of "inhaltsgleich" (of equal content), §18, etc., are certainly nearer the original than the expressions used by the French translator. Where custom has established a translation of a standard technical word, Professor Townsend generally appears to have used it, as, for example, "set" for "Menge," "enumerable" for "abzählbar," "(number) field" for "(Zahl) körper," to all of which excellent alternate expressions are, however, in common use. The expression "numerical multiplicity," translated on page 126 from the French "multiplicité numérique," suffers on the other hand by comparison with Ziwet's† "manifold of numbers" as an equivalent for the German word "Zahlenmannigfaltigkeit."

* To illustrate: Hilbert, in his lectures, used effectively a geometry in which "points" are the ordinary positive integers, and "lines" are the ordinary negative rational numbers. The principle involved was, of course, used before Hilbert by others, but it has taken a far more tangible form in Hilbert's hands. See Poincaré, l. c., pp. 250, 252, etc.

† In his translation of Sommer's review above mentioned.

Professor Halsted * informs us that the translation of "Anordnung" *must* be "arrangement," but the reviewer feels that several other words are also suitable translations of "Anordnung," in its technical sense, just as "ausgezeichnet," when used technically, need not necessarily be rendered "remarkable." † The translations "order," "arrangement," and "order of sequence" are to be found in Professor Townsend's edition, and any one of them, used consistently, would have been acceptable. Professor Ziwet, in the review mentioned above, uses "order." Perhaps even a still better word might be found, but Professor Townsend is to be congratulated in not having followed the French translator in his unhappy choice. The omission of the heading "Erklärung" can be defended, although the reviewer personally inclines to its retention.

The work is unfortunately somewhat marred by several errors, some of which, occurring at critical points, are apt to be misleading. On page 24, line 11, the word "all" is wholly wrong, as the translator might have known from Hilbert's lectures or at least from Dr. Sommer's review. It may be admitted that the original is not entirely clear on this point, but no such word as "all" occurs in the German edition. The translation of "einzig" by "definite" in line 8, page 53, may possibly be confusing; *any* single point is to determine a segment O . The word "only" must be inserted between "which" and "such" on line 11 from the bottom of page 98. On page 107, line 1, we are not to understand that Pascal's theorem is *identical* with the commutative law, but merely that the truth of the commutative law insures that Pascal's theorem holds, under the given conditions. ‡ On page 124, line 11, we understand that the extraction of the square root of f_3 is the third in order, *not* that it is now to be performed "a third time." The statement on page 125, that "every regular polygon can be

* In a brief review of Professor Townsend's edition, *Science*, August 22, 1902. The review is not especially appreciative and dwells on several mistakes and misprints. Some of the rather harsh criticisms are quite unfounded, as, for instance, those regarding axiom IV, 1, p. 12; theorem 16, p. 22; and a sentence on page 25, lines 26-28; while others seem trivial, for example, that of "emanating," p. 13, etc. The errors on pp. 24 and 125, noticed by Professor Halsted, and the omission of the heading "Erklärung," are considered later in the present review. The remainder of Professor Halsted's criticisms seem unimportant, in the sense that the passages in question are at least never misleading as they stand, and certainly do not call for separate reconsideration here.

† An accepted translation for "ausgezeichnete Untergruppe" is "self-conjugate subgroup."

‡ See Poincaré, l. c., p. 267.

constructed by the drawing of straight lines and the laying off of segments" will scarcely mislead any one. It is conceivable that this curious mistranslation was caused by some oversight, but it seems that "jene" has been mistaken for "jede(s)": "Every regular polygon" should, of course, be replaced by "the above-mentioned regular polygons."

But the most dangerous errors occur in the translation of the important new appendix of the French edition. On page 127, lines 17, 19, 20, 21, "par" is translated "from" instead of "by," and on page 130, line 11, "vis-a-vis de" is also rendered "from" instead of "with respect to," while on page 128, line 9, "imposée" is changed into "confined." A reading of the paragraphs in question will disclose the fact that at least the first two of these mistakes totally destroy all meaning at critical points. These errors, and some minor ones, make this conclusion the poorest part of the English edition.

The translator's English sometimes lacks the true idiomatic ring, the worst instances being, perhaps, the footnote on page 85, and the sentence which begins on line 7, page 124. There are also several minor faults of translation, which will fortunately not mislead the student. Among these are changes from singular to plural nouns and *vice versa*, for instance, "square root(s)," line 23, page 121; the insertion and omission of words, as "consequently" line 21, page 127; and mistranslations of words. Of such minor errors the reviewer has noticed, not counting duplicates, nineteen.

Among the misprints found only two are serious: the reference to "§ 30" on page 98, line 24, should be "§ 27"; and "and" three lines from the bottom of page 123, should be "any." The misprint (or error) in the original in formula (1), § 33, corrected in the Errata (page 111) of the French edition, is corrected properly on page 104 of Professor Townsend's translation.

The word "all" on page 74, line 22, while not strictly a mistranslation of language, gives rise to an impression which is faulty, and which is not intended in the original. For it is nowhere proved that the assumption of each of the axioms IV, 1-5, is necessary for the proof of Desargues's theorem. The proper impression would possibly be conveyed by substituting the phrase "or the entire set of axioms of congruence" for the corresponding phrase in the text; the idea being that, whereas we have assumed axioms IV, 1-5, it is *necessary* to *add* to

these axiom IV, 6, but it is not shown that we need *retain* each of the axioms IV, 1-5.

The notation for a segment joining two points, *e. g.*, 0 and $\frac{1}{2}$, has been changed in § 17, perhaps for the better, from $0\frac{1}{2}$ to $(0, \frac{1}{2})$. There are also other slight changes of notation, which are generally commendable. But a great deal is lost by not following Hilbert's italicization of words and of some of the theorems, which enabled him to emphasize what he considered most important. In particular the words "NON PAS" of the principal theorem of the new conclusion to the French edition, which Hilbert and Laugel emphasized not only by extraordinary type, but also by extraordinary position in the sentence, are rendered on the last line of page 128, by an ordinary "not," in ordinary type.

The Open Court Publishing Company deserves praise for continuing to publish translations of foreign scientific classics into English. The book is well printed, except for a few bad figures, of which the worst are figures 18 (page 41) and 38 (page 76), the latter being actually misleading. The type is good, excepting the identity sign. With regret we are compelled to notice, however, that the edition is not wholly satisfactory on account of the errors mentioned above. In its present form it can scarcely be recommended to students unless they can read neither German nor French; and then it should be used only under competent guidance. But most of the serious errors could be corrected by changing a few words, and if a page of such corrections were inserted in the volume it could be used by students with greater safety.

To other than university graduate students *no* edition of the book appeals. For the statement* that it has pedagogical value, or that it is to be of influence on elementary instruction, only means that students of very advanced mathematics, by their knowledge of it, may reflect into elementary books or teaching which they may influence, something of its general spirit, almost nothing of its actual contents. To insert the system of axioms proposed by Hilbert in an elementary (high school) book or, for instance, to try to introduce into such a text-book the theory of motion as based upon congruence, or even the proof that all right angles are equal, would be as ill considered as to expect the average high school teacher to grasp the meaning of the original in its entirety.

* Sommer, l. c., p. 299; Townsend, preface to English edition.

We must rejoice, however, in proof of the wide circulation of Hilbert's ideas, that both a French and an English translation have actually been published. A widely diffused knowledge of the principles involved will do much for the logical treatment of all science and for clear thinking and clear writing in general.

E. R. HEDRICK.

YALE UNIVERSITY,
September, 1902.

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Compliments of the author.

ON THE SUFFICIENT CONDITIONS IN THE
CALCULUS OF VARIATIONS.

BY DR. E. R. HEDRICK.

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ON THE SUFFICIENT CONDITIONS IN THE CALCULUS OF VARIATIONS.

BY DR. E. R. HEDRICK.

(Read before the American Mathematical Society, December 28, 1901.)

THE sufficient conditions in the calculus of variations have recently received a great deal of attention ; * and it would seem fitting that attempts be made to simplify their discussion whenever possible, and to render the agreement more exact between the known necessary and the known sufficient conditions. Such is the purpose of this paper, which also seeks to present the sufficient conditions in compact form. The work will to a large extent follow lectures delivered at Göttingen by Professor Hilbert, 1899-1901.

1. HILBERT'S INVARIANT INTEGRAL.† WEIERSTRASS'S SUFFICIENT CONDITION.

Let us consider a simple definite line integral

$$(1) \quad I = \int_{x_0}^{x_1} f(x, y, y') dx,$$

where

$$y' = \frac{dy(x)}{dx},$$

and where f is an analytic function of the three arguments x, y, y' , in a certain region R . Let us then restrict ourselves to the consideration of curves of integration contained in a realm B , consisting of curves of the type

$$(2) \quad y = \phi(x),$$

* Du Bois-Reymond, *Math. Annalen*, 15 ; Kneser, *Lehrbuch der Variationsrechnung*, and many memoirs in *Math. Annalen* ; Osgood, *Annals of Math.*, 2d ser., vol. 2, no. 3, and *Transactions Amer. Math. Soc.*, vol. 2, pp. 166, 273 ; Whittemore, *Annals of Math.*, 2d ser., vol. 2, no. 3 ; Bolza, *Ithaca Colloquium* (summer meeting, Amer. Math. Soc., Aug., 1901), unpublished, and *Transactions Amer. Math. Soc.*, vol. 2, p. 422 ; Weierstrass, lectures at Berlin, 1879-1882, unpublished ; Hilbert, lectures at Göttingen, 1899-1901, unpublished, etc.

† Compare Osgood, *Annals of Math.*, l. c., where Hilbert's proof is given.

where $\phi(x)$ together with its first derivative is a single valued and continuous function of x in the interval $x_0 \leq x \leq x_1$, and where $\phi(x_0)$ and $\phi(x_1)$ have certain fixed values y_0 and y_1 respectively. If the value of the integral I taken along a curve

$$(3) \quad y = y(x)$$

of the realm B be less than the value of I taken along any other curve of the realm B ,

$$(4) \quad y = Y(x) \equiv y(x) + \eta(x),$$

where $Y(x)$ is restricted by the same conditions as y and also by

$$(5) \quad |\eta(x)| < \delta, \quad |\eta'(x)| < \delta, \quad x_0 \leq x \leq x_1,$$

δ being a positive constant chosen at pleasure, then the curve (3) is said to render the integral I a *weak* minimum, as compared to curves of the realm B . If in place of the conditions (5) we merely require that

$$(6) \quad |\eta(x)| < \delta, \quad x_0 \leq x \leq x_1,$$

then the curve (3) is said to render I a *strong* minimum.* If in addition to the condition (5) [or (6)] we require that

$$(7) \quad \begin{aligned} \eta(x_i) &= 0; \quad x_0 + (i-2) \cdot \delta \leq x_i \leq x_0 + (i-1)\delta \leq x_1; \\ (i &= 2, 3, \dots, n, (n+1); \quad n \cdot \delta = x_1 - x_0), \end{aligned}$$

then we will say that the curve (3) renders I a *limited weak* [or *strong*] minimum.†

It can be shown ‡ that if the curve (3) is to render I a minimum (of any kind), $y(x)$ must satisfy Lagrange's equation

$$(8) \quad \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} = 0.$$

The solutions of this equation are called *extremals*.

Let us now assume: §

* For the above, cf. Osgood, *Annals of Math.*, l. c., p. 106.

† This is what Hilbert calls "ein Minimum bei stückweiser Variation."

‡ See Whittemore, l. c., p. 130, where Hilbert's proof is reproduced.

§ Compare Osgood, *Annals of Math.*, l. c., p. 113; *Transactions Amer. Math. Soc.*, l. c., p. 168.

(a) That an extremal C ,

$$(9) \quad y = y(x),$$

can be obtained, such that $y_0 = y(x_0)$ and $y_1 = y(x_1)$, i. e., such that C passes through the two fixed points $P: (x_0, y_0)$ and $Q: (x_1, y_1)$ given in the problem.

(b) That a one parameter family of extremals

$$(10) \quad y = \phi(x, \alpha)$$

can be obtained, where ϕ is an analytic* function of x and α , $x_0 \leq x \leq x_1$, $|\alpha| < \epsilon$; and where

$$(11) \quad \phi(x, 0) \equiv y(x).$$

(c) That

$$\frac{\partial \phi(x, 0)}{\partial \alpha} \neq 0, \quad x_0 < x \leq x_1.$$

It then follows that through any point of a suitably chosen neighborhood N about the curve C , one and only one extremal of the family (10) can be passed; and that (10) can be solved for α in terms of x and y ,

$$(12) \quad \alpha = \psi(x, y),$$

where $\psi(x, y)$ is a singly valued analytic function of x and y , in the neighborhood N . Hence also

$$y' = \frac{\partial \phi(x, \alpha)}{\partial x}$$

can be expressed as a singly valued analytic function of x and y ,

$$(13) \quad y' = p(x, y),$$

in the neighborhood N .

Hilbert now considers the integral

$$(14) \quad J = \int_{x_0}^{x_1} [f(x, y, p) + (y' - p)f_p(x, y, p)] dx$$

taken along any curve $\overline{C}$, $y = Y(x)$, joining P and Q , and

* Since f has been assumed analytic in x, y, y' , it follows that the coefficients in (8) are analytic, and hence that the solutions of (8) are analytic.

lying in the neighborhood N , where y and y' refer to the curve of integration and p denotes the above function $p(x, y)$, and where f_p means $\partial f / \partial p$. It is then shown that this integral J is independent of the path of integration,* i. e., independent of $Y(x)$. It is seen at once that $J_C = I_C$, where J_C denotes the value of the integral J taken along the curve C , and so on. But $J_{\bar{C}} = J_C$ since J is independent of the path. Hence the necessary and sufficient condition for a minimum, which is that $I_{\bar{C}} > I_C$, becomes at once $I_{\bar{C}} > J_{\bar{C}}$, or $(I - J)_{\bar{C}} > 0$, or

$$(15) \quad \int_{x_0}^{x_1} E(x, y, y', p) dx \Big|_{\bar{C}} > 0,$$

where $\bar{C}$ is any curve in N other than C , and where

$$(16) \quad E(x, y, y', p) \equiv f(x, y, y') - f(x, y, p) - (y' - p)f_p(x, y, p).$$

It follows that

$$(17) \quad E(x, y, y', p) \geq 0, \quad x_0 \leq x \leq x_1$$

for all x, y near the curve C , for the function p found above, and for any y' whatever, is a sufficient condition for a strong minimum,† where the sign of equality is to hold only along the extremals (10). And, moreover, the condition

$$(18) \quad E(x, y, y', p) \geq 0, \quad x_0 < x < x_1,$$

for all x, y, p on C and for all y' whatever, is a necessary condition for a strong minimum.† For if, for simplicity, the family of extremals (10) be so taken as all to pass through $P: (x_0, y_0)$, then a curve can be found which renders the integral less than C does, in case $E < 0$ at any point of C in any direction $y' \neq p$. For we need only take as the comparison curve desired an extremal other than C , together with a curve which cuts C at this point in the given direction. The integral $\int E dx$ taken along such a curve is certainly negative.

Let us now consider a fixed point $R: (x, y)$, on the curve C , and $f(x, y, y')$ as a function of y' alone, for these fixed values of x and y . Let us then draw in a $y'f$ -plane the curve

* See Osgood, *Annals of Math.* l. c., p. 122.

† Provided, of course, that conditions (a), (b), (c) hold.

$f=f(y')$. By (16) the inequality $E > 0$ becomes

$$(19) \quad \frac{f(y') - f(p)}{y' - p} \leq f_p(p) \text{ according as } y' - p \leq 0.$$

Let θ be the angle made by the axis of y' with the tangent to the $y'f$ -curve at the fixed point where $y' = p$. And let ψ be the angle between the y' axis and the chord joining this fixed point to a variable point where $y' = y'$. Then, as is easily seen from a figure,

$$\tan \theta = f_p(p) \quad \text{and} \quad \tan \psi = \frac{f(y') - f(p)}{y' - p}.$$

Hence the inequality (19) becomes

$$(20) \quad \tan \psi \leq \tan \theta \quad \text{according as} \quad y' \leq p.$$

But this is surely fulfilled (except, of course, for $y' = p$) provided the $y'f$ -curve is always concave upward, the f axis having been taken vertical, for the range of values of y' considered.

Hence the condition

$$(21) \quad \frac{\partial^2 f}{\partial y'^2} > 0,$$

for all x, y on the curve C and for all y' considered, is a sufficient condition for a minimum.* For if (21) is satisfied, the $y'f$ -curve for every point (x, y) , on or near $\dagger C$, will be concave upwards, for the whole range of values of y' considered, and hence, by the above, the Weierstrass sufficient condition (17) will be satisfied.†

The condition (21) is, however, by no means necessary. A necessary condition usually attributed to Legendre resembles it in form. This necessary condition requires that $\frac{\partial^2 f}{\partial y'^2} \geq 0$ for all

* Weak or strong, according as the values of y' are restricted, or not. Here, as elsewhere, the region R of page 11 must not be overstepped. In particular, the value $y' = \infty$ must always be carefully investigated.

† On account of the continuity of $\frac{\partial^2 f}{\partial y'^2}$ in x, y, y' .

‡ This also follows from $\text{Limit}_{y'=p} \left[\frac{E(x, y, y', p)}{(y' - p)^2} \right] = \frac{\partial^2 f}{\partial y'^2} \Big|_{y'=p}$.

x, y on the curve C and for $y' = p$.* A proof of this theorem is given in the following section.

2. LIMITED VARIATION. LEGENDRE'S CONDITION.

If we now consider limited variation only, we have the following theorem: *Lagrange's, Legendre's narrower,† and Weierstrass's ‡ conditions are together sufficient for a limited strong minimum.*

We suppose first found a solution C of Lagrange's equation connecting the points P and Q . Around any point of C a region can so be bounded off that one and only one extremal exists joining the given point to any point of the region; which follows at once from the Cauchy-Kowalewski existence theorems,

since $\left. \frac{\partial^2 f}{\partial y'^2} \right|_C \neq 0$ at such a point, from the narrower condition of

Legendre. If now we require that $\eta(x) = 0$ at least once in the interior of each of a set of regions such that the Cauchy-Kowalewski solutions starting from the one end point are unique up to the other end point, we surely have in each such region a satisfactory field for the Weierstrass-Hilbert theory. It follows that the Weierstrass-Hilbert reasoning can be applied, and hence, if the Weierstrass condition is satisfied, the curve C actually renders the integral I a limited strong minimum.

Likewise it is clear that *Lagrange's, and Legendre's narrower conditions alone are together sufficient for a limited weak minimum.* For Legendre's condition, $\left. \frac{\partial^2 f}{\partial y'^2} \right|_C > 0$ is satisfied. But this insures § the fulfillment of Weierstrass's sufficient condition, by the end of Section 1, since $\frac{\partial^2 f}{\partial y'^2}$ is a continuous func-

* For this condition we shall use the notation $\left. \frac{\partial^2 f}{\partial y'^2} \right|_C \geq 0$; and we shall call it, temporarily, Legendre's *broadier* condition; later, Legendre's *necessary* condition. The corresponding condition, $\left. \frac{\partial^2 f}{\partial y'^2} \right|_C > 0$, where the sign of equality may not hold, we shall call, temporarily, Legendre's *narrower* condition; later, *sufficient*. It is readily seen that $\left. \frac{\partial^2 f}{\partial y'^2} \right|_C > 0$ is sufficient for a weak minimum, under conditions (a), (b), (c).

† See preceding footnote.

‡ See condition (17) above.

§ For weak variation, of course.

tion of x, y, y' . On account of this theorem we will henceforth call the condition $\left. \frac{\partial^2 f}{\partial y'^2} \right]_C > 0$ *Legendre's sufficient condition*.

We will now proceed to show that Legendre's broader condition is necessary for any kind of a minimum whatever. We have seen that Legendre's sufficient condition, $\left. \frac{\partial^2 f}{\partial y'^2} \right]_C > 0$ is, together with Lagrange's condition, sufficient for a limited weak minimum. But that I have a *minimum* is equivalent to saying that $-I$ has a *maximum*. Hence $\left. \frac{\partial^2 f}{\partial y'^2} \right]_C < 0$ is, together with Lagrange's condition, a sufficient condition for a limited weak *maximum*. Given now a solution of Lagrange's equation, C , joining P and Q , if $\left. \frac{\partial^2 f}{\partial y'^2} \right]_C < 0$ at any point of C , then $\left. \frac{\partial^2 f}{\partial y'^2} \right]_C < 0$ for a whole interval, since $\frac{\partial^2 f}{\partial y'^2}$ is a continuous function of x, y, y' , and C is a continuous curve whose tangent varies continuously. But since $\left. \frac{\partial^2 f}{\partial y'^2} \right]_C < 0$ is a *sufficient condition* for a limited weak *maximum*, it follows that *any limited weak variation* in this interval gives us a curve for which I has a *less value* than along C . It follows that Legendre's broader condition, $\left. \frac{\partial^2 f}{\partial y'^2} \right]_C \geq 0$, is a *necessary condition* for a minimum of any kind whatever; since otherwise we have actually found a comparison curve C' such that $I_{C'} < I_C$, and this comparison curve occurs in any possible choice of realms of curves to be considered since it is a *limited weak variation* of C , the tangents to which can be made, if necessary, to turn continuously.* On account of this theorem we shall henceforth call the condition $\left. \frac{\partial^2 f}{\partial y'^2} \right]_C \geq 0$ *Legendre's necessary condition*.

It seems fitting at this point to call attention to the general usefulness of the idea of the limited variation, which has led us so easily to Legendre's necessary condition. The question

* Or even analytically, by the now well known method originally due to Schwarz.

of geodetics in the surface theory, and many questions in mechanics, notably the Hamilton-Jacobi theory, require only the use of *limited* variation, and indeed it is peculiarly suited to the needs of these and other similar subjects. Its greater simplicity in statement and treatment commend it above methods commonly employed, and its results are more easily obtained and grasped. For instance, in the surface theory, any geodetic line actually renders the integral of length on the surface a *limited* strong minimum along *any* portion of it which joins *any* two points upon it.* This statement brings into a clear light the essential characteristics of a geodetic as a "shortest" line, and its application to such special surfaces as the sphere and the anchor ring are particularly vivid.

3. UNLIMITED VARIATION. PRODUCTION OF JACOBI'S CONDITION FROM WEIERSTRASS'S CONDITION.†

Jacobi's condition was originally erroneously stated as a sufficient condition, and was shown by Weierstrass to be insufficient for strong minima. Thus far we have obtained Weierstrass's condition (Section 1) as sufficient for a strong minimum under the necessary conditions (a), (b), (c) of Section 1. We now propose to obtain Jacobi's condition as a necessary condition for any (*unlimited*) minimum from consideration and elimination of these conditions (a), (b), (c). We shall then show that Jacobi's condition (together with Lagrange's and Legendre's) is a *sufficient* condition for *weak* minima, the Weierstrass condition being unnecessary. Thus the Jacobi condition precisely covers the introduction of the *unlimited* variation, for exactly analogous results have been obtained for *limited* variation (Section 2) with its omission.

We will suppose the Lagrange condition satisfied, since it is a *first* necessary condition for any discussion. Let the supposed solution C of our problem be the curve

$$(1) \quad y = y(x)$$

joining $P: (x_0, y_0)$ and $Q: (x_1, y_1)$. The Lagrange equation

$$(2) \quad y'' \frac{\partial^2 f}{\partial y'^2} + y' \frac{\partial^2 f}{\partial y \partial y'} + \frac{\partial^2 f}{\partial x \partial y'} - \frac{\partial f}{\partial y} = 0$$

* In this particular theorem, of course, no new *fact* is involved.

† In connection with this section, see Osgood, *Transactions Amer. Math. Soc.*, l. c., p. 166.

has in general a two-parameter family of solutions

$$(3) \quad y = \phi(x, c_1, c_2)$$

where ϕ is an analytic function of x, c_1, c_2 , if, as supposed, f is analytic in x, y, y' .

Let us now suppose that c_1 and c_2 are so chosen that

$$(4) \quad \phi(x, 0, 0) \equiv y(x),$$

so that for $c_1 = 0, c_2 = 0$, our curve C results. Further let $c_1 = t\gamma_1$ and $c_2 = t\gamma_2$. Then (3) becomes

$$(5) \quad y = \phi(x, t\gamma_1, t\gamma_2).$$

Since ϕ is analytic in c_1 and c_2 we have, when γ_1 and γ_2 are regarded as constant,

$$(6) \quad \begin{cases} y = \phi(x, t\gamma_1, t\gamma_2) = \Phi(x, t), & \text{say,} \\ = \Phi(x, 0) + t \frac{\partial \Phi(x, 0)}{\partial t} + \frac{t^2}{2!} \frac{\partial^2 \Phi(x, 0)}{\partial t^2} + \dots \\ = y(x) + t\phi_1(x) + t^2\phi_2(x) + \dots \end{cases}$$

which converges uniformly for sufficiently small values of t .

Now, regarding γ_1 and γ_2 as constant, and (6) as our one-parameter family of solutions required by condition (b), Section 1, let us set

$$(7) \quad p = \frac{\partial \Phi(x, t)}{\partial x} = \frac{dy(x)}{dx} + t \frac{d\phi_1(x)}{dx} + t^2 \frac{d\phi_2(x)}{dx} + \dots$$

If now we can solve (6) for t as a single valued function of y and x , we can insert the value found in (7) and have p defined as a single valued function of y and x , *i. e.*, of the position; and this will hold for sufficiently small values of t , since the series used are then uniformly convergent, *i. e.*, it will hold for all points in a suitably chosen neighborhood of C . Such a solution will then satisfy the purposes of the Weierstrass-Hilbert theory. But for this (compare condition c, Section I) it is only necessary that

$$(8) \quad \phi_1(x) \neq 0.$$

But

$$\phi_1(x) = \left. \frac{d\Phi(x, t)}{dt} \right]_{t=0},$$

whence

$$(9) \quad \phi_1(x) = \frac{\partial \phi(x, c_1, c_2)}{\partial c_1} \frac{dc_1}{dt} + \frac{\partial \phi(x, c_1, c_2)}{\partial c_2} \frac{dc_2}{dt}$$

$$= \frac{\partial \phi}{\partial c_1} \gamma_1 + \frac{\partial \phi}{\partial c_2} \gamma_2 = \eta_1 \gamma_1 + \eta_2 \gamma_2,$$

$$\text{where } \eta_1 \equiv \frac{\partial \phi}{\partial c_1}, \quad \eta_2 \equiv \frac{\partial \phi}{\partial c_2}.$$

If then we can so choose γ_1 and γ_2 that $\phi(x)$ does not vanish between x_0 and x_1 the Weierstrass-Hilbert theory can be applied.

And further, the possibility of such a choice is necessary for the existence of a minimum. For, consider the family of extremals out of (3) which pass through (x_0, y_0) and suppose that γ_1 and γ_2 are so chosen that (6) represents this family.* If then $\phi_1(x)$ vanishes at any point R between x_0 and x_1 then the equations

$$(10) \quad \begin{cases} y = \Phi(x, t), \\ \frac{d\Phi(x, t)}{dt} = 0, \\ y = y(x) \equiv \Phi(x, 0) \end{cases}$$

are satisfied at this point, since

$$\left. \frac{d\Phi(x, t)}{dt} \right]_{t=0} = \phi_1(x).$$

Hence the envelope of the above family of extremals through (x_0, y_0) cuts the curve C at any such point as R .

If now this envelope be a single point, then compare the value I_C with $I_{\bar{C}}$ where $\bar{C}$ is any other extremal of the family. Since J is independent of the path and since J reduces to I for all extremals, it follows that C does not render I a proper (unlimited) minimum of any sort whatever, between P and R or

* It can be shown from the considerations at the end of this Section, that this is the best possible choice.

between P and any point beyond R . For $I_C]_P^R = I_{\bar{C}}]_P^R$ and $\bar{C}$ is an analytic weak variation of C .

If on the other hand the envelope is a curve, then compare $I_C]_P^R$ with $I_C]_P^{R'} + I_E]_{R'}^R$, where $\bar{C}$ is any other extremal of the family, E the envelope, and R' the point where $\bar{C}$ meets E . Since the envelope is tangent at every point to an extremal of the family, it follows, as before, that $J_E = I_E$ and hence $I_C]_P^R = I_{\bar{C}}]_P^{R'} + I_E]_{R'}^R$. Hence C does not render I a proper * (unlimited) minimum between P and R nor between P and any point beyond R .

It follows that if we can so choose γ_1 and γ_2 that

$$\phi_1(x) \neq 0, \quad x_0 < x \leq x_1,$$

the Weierstrass-Hilbert theory can be applied; and further the possibility of such a choice is a necessary condition for the existence of an (unlimited) minimum of any sort.

We know that

$$(11) \quad y = \phi(x, c_1, c_2) = \Phi(x, t)$$

must satisfy Lagrange's equation

$$(12) \quad L(y) \equiv \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} = 0$$

or

$$(13) \quad 0 = L(\Phi(x, t)) = L(y(x) + t\phi_1(x) + t^2\phi_2(x) + \dots)$$

$$= L(y(x)) + \frac{dL(y(x) + t\phi_1(x) + \dots)}{dt} \Big|_{t=0} t + \frac{d^2L}{dt^2} \Big|_{t=0} \frac{t^2}{2!} + \dots$$

for all sufficiently small values of t . Hence the coefficient of t must vanish separately,

$$(14) \quad \frac{d}{dt} \{L(y(x) + t\phi_1(x) + t^2\phi_2(x) + \dots)\} \Big|_{t=0} = 0.$$

But $L(\xi)$ involves $\xi, \frac{d\xi}{dx}, \frac{d^2\xi}{dx^2}$. Hence we have

* It can indeed be shown that C does not even render I an improper minimum, for the envelope cannot be an extremal. It is to be noticed that all the above work holds even when the extremals cut the envelope.

$$(15) \quad \frac{\partial L(\xi)}{\partial \frac{d^2 \xi}{dx^2}} \frac{d^2 \xi}{dx^2} \frac{d}{dt} + \frac{\partial L(\xi)}{\partial \frac{d \xi}{dx}} \frac{d \xi}{dx} \frac{d}{dt} + \frac{\partial L(\xi)}{\partial \xi} \frac{d \xi}{dt} \Big]_{t=0} = 0,$$

where

$$\xi \equiv y(x) + t\phi_1(x) + t^2\phi_2(x) + \dots;$$

or

$$(16) \quad \frac{\partial L(y)}{\partial y''} \phi_1''(x) + \frac{\partial L(y)}{\partial y'} \phi_1'(x) + \frac{\partial L(y)}{\partial y} \phi_1(x) = 0,$$

an equation which must be satisfied by $y = y(x)$. Using the definition of $L(y)$, we find by a simple calculation that $\phi_1(x)$, and hence both η_1 and η_2 , must satisfy the following linear differential equation of the second order :

$$(17) \quad f_{y'y'} \cdot \eta'' + \frac{df_{y'y'}}{dx} \eta' + \left(\frac{df_{y'y'}}{dx} - f_{yy'} \right) \eta = 0,$$

where

$$\eta'' \equiv \frac{d^2 \eta}{dx^2}, \quad f_{y'y'} \equiv \frac{\partial^2 f}{\partial y'^2}, \quad \text{etc.,}$$

as usual. But (17) is precisely Jacobi's equation ; and Jacobi's whole condition consisted in finding whether an integral of this equation

$$\phi_1(x) = \gamma_1 \eta_1 + \gamma_2 \eta_2$$

existed, which did not vanish between x_0 and x_1 . Hence the above are precisely Jacobi's conditions, except that the equality sign in our condition must be omitted in the usual discussion, and the corresponding case made a subject of further investigation.

The rest of the results announced at the beginning of the article now follow readily, and will be left to the reader.

4. SUMMARY OF CONDITIONS.

Collecting our results, we may say that the following conditions hold :

	Limited Variations.		Unlimited Variations.	
	Weak.	Strong.	Weak.	Strong.
<i>Necessary.</i>	Lagrange's, Legendre's necessary.	Lagrange's, Legendre's necessary, Weierstrass's necessary.	Lagrange's, Legendre's necessary, Jacobi's.	Lagrange's, Legendre's necessary, Jacobi's, Weierstrass's necessary.
<i>Sufficient.</i>	Lagrange's, Legendre's sufficient.	Lagrange's, Legendre's sufficient, Weierstrass's sufficient.	Lagrange's, Legendre's sufficient, Jacobi's.	Lagrange's, Legendre's sufficient, Jacobi's, Weierstrass's sufficient.

It is seen on glancing at the table that from the simple conditions (Lagrange's and Legendre's) for limited weak variation we proceed to any other case by adding Weierstrass's conditions in the case of a strong minimum, and Jacobi's in case of an unlimited minimum, only. It is to be hoped that advances may be made in bringing the necessary and sufficient conditions more closely together or into entire coincidence. The above table represents *substantially* the present known conditions, in the belief of the writer.

In special problems the irksomeness of these conditions can sometimes be circumvented. For instance, given a problem in which $\frac{\partial^2 f}{\partial y'^2} > 0$ for all values of x, y, y' , then the necessary and sufficient condition for a limited strong minimum is the possibility of finding a solution of Lagrange's equation joining the two given end points. Such is the case in the geodetic problem and also in the integral which leads to Hamilton's principle; and in each of these cases, fortunately, a limited strong minimum is all that is desired. Similar simplification occurs in every case when $\frac{\partial^2 f}{\partial y'^2} > 0$ for all x, y, y' . For then Legendre's and Weierstrass's conditions are always satisfied, and may be abstracted from the above table. For this reason Hilbert has called a problem in which $\frac{\partial^2 f}{\partial y'^2} > 0$ for all x, y, y' contained in a singly connected region R , in which the given end points lie, a "regular" problem of the calculus of variations.

5. CONCLUSION. HILBERT'S EXISTENCE THEOREM.

A misunderstanding has sometimes arisen in regard to Hilbert's existence proof. Its purpose is not to establish further conditions nor to solve more general problems, but is to reassure us as to the possibility of satisfying the conditions already found, in the more favorable cases. We have seen that a *necessary* condition for any minimum was the possibility of finding a solution of Lagrange's equation joining the two end points P and Q ; and that this is a *sufficient* condition for a limited strong minimum, in a *regular* problem. That such a solution exists, even in the most favorable cases, is by no means certain from the ordinary theory of differential equations. It was the original purpose of Hilbert's proof (merely) to show that, in this most favorable case of a regular problem, a curve must exist which renders our integral an unlimited strong minimum, compared with all continuous comparison curves; and that the minimizing curve is composed of a finite number of pieces of extremals. Hilbert's existence theorem may therefore properly be called a theorem in differential equations.

At present the results cannot be said to demonstrate more than that a curve exists which renders the integral an *improper* minimum. This result *cannot* be extended in general, and it remains to show that Lagrange's equation is necessary, not only for proper, but also for improper minima.

In conclusion, the author desires to enter protest against the extreme complication recently introduced in some quarters into the essentially simple subject of the calculus of variations. In the case of the only modern text-book on the theory,* this condition is so exaggerated as to essentially mar the usefulness of the book, in that many who would otherwise interest themselves in the subject, are repelled by the style and treatment. It is to be sincerely regretted, in the opinion of the writer, that this tendency has been followed in some of the recent memoirs.

YALE UNIVERSITY,
June, 1902.

* Kneser, Variationsrechnung.



Alexander Ziwet

**SUPPLEMENTARY NOTE ON THE CALCULUS OF
VARIATIONS.**

THE DOCTRINE OF INFINITY.

BY DR. E. R. HEDRICK.

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10

SUPPLEMENTARY NOTE ON THE CALCULUS OF VARIATIONS.

BY DR. E. R. HEDRICK.

IN a recent paper in the BULLETIN,* the present writer stated the theorem: "The condition $\partial^2 f / \partial y'^2 > 0$ for all x, y on the curve C , and for all y' considered, is a sufficient condition for a minimum" (of the integral $\int f(x, y, y') dx$ along the curve C), provided that certain preliminary requirements are satisfied. This theorem stands in apparent contradiction with an example given by Professor Bolza;† and while the contradiction is *only* apparent, it seems fitting to point out clearly the actual agreement of my results with those of Professor Bolza.

In the article mentioned I have required (page 11) that the integral $\int f(x, y, y') dx$, shall be considered only in a region R (of the x, y, y' space), such that the integrand $f(x, y, y')$ is

* "On the sufficient conditions in the calculus of variations," BULLETIN, vol. 9 (2), no. 1 (Oct., 1902), p. 15.

† "Some instructive examples in the calculus of variations," BULLETIN, vol. 9 (2), no. 1 (Oct., 1902), Example II, p. 9.

analytic at each point of R , *i. e.*, for all sets of values of x , y , y' in R . A region is here a closed manifold of points, *i. e.*, it possesses a boundary, so that a point upon the boundary of R belongs to the region. Hence for the reasoning throughout it is necessary to choose such a region R . It is true that this restriction is by no means necessary for most of the results of the paper, but without it some of the proofs, and in particular the above theorem, would have to be modified. It is to be noticed, however, that we may choose an *unbounded* region R_1 (*i. e.*, exclusive of the boundary) provided R_1 lies wholly within a region R such as is required above, and that all the results of the paper hold when the given integral is considered only for points *inside of* R_1 , or in particular *inside of* R itself. In other words, the results given hold in any region (inclusive or exclusive of the boundary) provided the points for which the integrand $f(x, y, y')$ is not analytic be excluded from the region (*i. e.*, from its interior and from its boundary).

In Professor Bolza's example, I would then, for the purposes of my paper, *first* choose as my region R the region

$$R: \quad |x| \leq A; \quad |y| \leq A; \quad |y'| \leq A.$$

Any positive number A having been chosen for the example once for all, the investigation of minima depends on our ability to find, *secondly*, a value for δ so small that the supposed solution renders the integral value less than along any other curve for which

$$|\eta(x)| < \delta, \quad x_0 \leq x \leq x_1.$$

It is seen at once that any *order of choice* of these quantities A and δ corresponds exactly to the same order of choice of the h and k of Professor Bolza's article, respectively; whereas the order of choice necessary for the considerations of my paper is exactly opposite to that made by Professor Bolza. For, from my standpoint, k cannot be *first* chosen and then kept fixed while h (*secondly*) approaches zero, as is necessary for Professor Bolza's argument. A choice of h *after* k has been chosen and fixed, necessitates a choice of $A \geq k/h$. This I would regard as the first step, after which I would have δ at my disposal, which would necessitate a re-choice of k , if δ be chosen less than k . It readily follows that, from my standpoint, the conclusion $\Delta I < 0$ cannot be drawn, and that the X axis actually renders

the integral in question a strong minimum,* in the above region R , i. e., in any region of values of x, y, y' from the interior and boundary of which the values $x = \infty, y = \infty, y' = \infty$ are excluded. Accordingly the example cited, as also Professor Bolza's theorem at the top of page 9, are perfectly reconcilable with my theorem quoted above.

The integrand in question is discontinuous for infinite values of either x or y or y' , and in fact becomes infinite to the second order for $y' = \infty$. It may be pointed out that the same apparent contradiction will arise in the case of any integral whose integrand becomes infinite to the second order for any value of y' (not necessarily $y' = \infty$); but again the contradiction is only apparent. It might be questioned whether simple ordinary integrals, such as the integral of length, $\int \sqrt{1 + y'^2} dx$, do not present similar phenomena, so that the results of my paper would be restricted in their application, in all ordinary examples, to a closed region (of the x, y, y' space) in which the integrand is analytic. Such are indeed the formal requirements of the paper. But it may be noted that if the transformed integral

$$\int f\left(x, y, \frac{1}{x'}\right) x' dy,$$

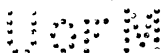
where $x' \equiv dx/dy$, be analytic for the directions previously excluded ($y' = a, x' = 1/a$), then the whole reasoning applies to the comparison curves previously excluded; by a simple process of transformation and subdivision of the curve, at least provided that the comparison curves in question satisfy the simple requirement of not being parallel to each axis an infinite number of times in any interval.

And further, the requirement made on page 11 is by no means necessary for the proof of most of the theorems. In view, however, of the aims of the article, it was deemed advisable to include this restriction, in order to simplify the proofs, and in order to arrive at the theorem mentioned above without too extended a discussion of possible alternate restrictions.

E. R. HEDRICK.

SHEFFIELD SCIENTIFIC SCHOOL,
November, 1902.

* The distinction between strong and weak minima holds as usual because the slope restriction $|y'| \leq A$ was made before any particular extremal was found, and does not regard any particular extremal.



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THE DOCTRINE OF INFINITY.

Die Grundsätze und das Wesen des Unendlichen in der Mathematik und Philosophie. Von Dr. K. GEISSLER. Leipzig, Teubner, 1902. 4to, 417 pp.

THE recent tendency toward critical investigation of the axioms underlying mathematical sciences has led to the necessity of discussing the philosophical basis of the whole mathematical structure, and it is to be hoped that more attention will be given the subject from a purely philosophical standpoint.

The announcement of a book treating the old stumbling block of the infinite from both the mathematical and the philosophical points of view was therefore of the greatest interest, and the character of the book was apparently guaranteed by the excellent reputation of the publishers. But the book—though not lacking in a certain kind of interest—is in several respects most surprising, and, as we shall see, somewhat disappointing.

The first 296 pages are devoted to mathematical investigations, and we shall consider these alone. A great variety of topics are treated, such as the parallel axiom, indeterminate forms, tangents, limits, derivates, velocity, curvature and many

other problems involving continuity. It is not here proposed to discuss these subjects in detail, but to confine ourselves to one serious matter, which the author regards as fundamental and which certainly affects the whole of his reasoning. This is his introduction of certain (fixed) infinitely great and infinitely small numbers (or segments in geometry).

There are two reasons for considering the subject seriously. In the first place it is common, in certain elementary treatises on the calculus, to introduce such infinitely small quantities and to deal with them under the name of "differentials." From the standpoint of mathematical pedagogy, it is certainly of vital importance to discover to what conclusions the introduction of such numbers leads, both with regard to the correctness of using them at all and with regard to the effect upon the student's conceptions and his mode of thought. Secondly, from the point of view of the foundations of mathematics, it is interesting to investigate the axioms tacitly assumed and to see just what axioms (of the usual set) may be retained and what new axioms need be added in the logical treatment of such infinitely great and infinitely small quantities, capable of assuming fixed constant values.

One of the axioms generally accepted in both arithmetic and geometry is that of Archimedes, which may be stated for arithmetic as follows: *Given any two positive numbers a and b , of which b is the greater, there exists a certain positive integer n , such that $a + a + a + \dots$ (n times) $> b$.*

Passing over the first few pages of the book, which are intended for very elementary students, we find, on page 35, in direct violation of Archimedes's axiom, the introduction of a certain positive ($\neq 0$) number δ , termed an "infinitely small" number, such that

$$\delta + \delta + \delta + \dots (n \text{ times}) < 1$$

for any (finite) integer n . And the "fundamental theorem" is repeatedly stated (e. g., page 141) as follows: "The product of a finite quantity and an infinitesimal is always infinitesimal"; while on page 66 it is stated that "the concept of an infinitesimal does not coincide exactly with zero." *

These statements explicitly deny the axiom of Archimedes, though the author does not mention the fact. Nor will the

* See also p. 46; and p. 90, where an infinitesimal unit is considered.

non-euclidean geometer protest against a geometry in which certain usual axioms are expressly violated, provided the fact of such violation is constantly kept in mind and no appeal whatever is made to ordinary geometry. In particular, one axiom which is at the basis of all considerations of continuity in the ordinary sense having been rejected, we naturally expect a thorough research of the axioms of continuity which are assumed to hold. We shall see that the author has unfortunately not adhered to such a programme.

No result of the book can then be accepted, even for a non-euclidean, non-archimedean, geometry. But it is interesting to note that some of the results obtained are such as actually do hold in the non-archimedean geometry actually constructed by Hilbert, in his *Grundlagen der Geometrie*.

If we consider as our realm of numbers all the rational functions of a certain parameter t , and define "equality," "greater than," and "less than" by the rule $f(t) >, =, < g(t)$ according as $f(t) - g(t) >, =, < 0$ for all sufficiently large positive values of t , then all the axioms of arithmetic (see Hilbert, page 26), except the axiom of Archimedes, are satisfied.*

We find, for instance, in the book in hand, on page 62, that "a ratio of two infinite [or infinitesimal] numbers of the same class is a finite number, * * * and a product of two infinite numbers of the same class is not finite, but is infinite [or infinitesimal] of another class." Arrangement into classes might be made by the rule that numbers of any one class satisfy, among themselves, the axiom of Archimedes. It is then readily seen that the above statement holds true in Hilbert's non-archimedean geometry. The two numbers

$$\frac{a_m t^m + a_{m-1} t^{m-1} + \dots + a_0}{b_n t^n + b_{n-1} t^{n-1} + \dots + b_0} \text{ and } \frac{c_r t^r + c_{r-1} t^{r-1} + \dots + c_0}{d_s t^s + d_{s-1} t^{s-1} + \dots + d_0}$$

are of the same class if $m - n = r - s$. Their quotient is of the class $(m - n) - (r - s) = 0$, and is hence "finite," i. e., in the same class with 1; while their product is of class $(m - n) + (r - s)$ which is neither zero nor equal to $(m - n)$ unless $m - n = 0$. This latter exceptional case corresponds to the statement (page 62): "The class of finite numbers is that in which the product and the ratio of any two numbers of the class again belongs to the class" — which holds for positive

*The form is slightly different from Hilbert's.

numbers. We are also informed that the choice of which class is to be considered finite is not at our disposal.

On page 177 the author says "the motion of sliding" — (presumably continuous motion in euclidean space) "is to be thought of not in general, but only for a certain class of segments at a time." The notion expressed practically amounts to saying that the complete (non-archimedean) geometry assumed does not permit of continuous motion in the ordinary sense. The author is at great pains elsewhere* to explain away the difficulty that a passage from "finite" to "infinite" [or to "infinitesimal"] numbers, for instance, can be affected *neither* by a sudden spring, nor yet *without* such a spring. That he has violated a fundamental axiom of continuity, and hence cannot expect his geometry to be continuous in the ordinary sense, is not made clear.

Again, on page 194, for example, the author is surprised to find himself in difficulties with such words as "continuous," "hole-less," etc., as applied to space. And he declares that a segment (in geometry) or a number (in arithmetic), when added to a (greater) second, "does not alter the value of the second except when it is of the same class." And on the next page (195): "for this reason and only for this reason we shall be correct if we consider in any discussion the numbers (or segments) of any one class." But this is returning to archimedean geometry, and relinquishing our previous violation of the archimedean axiom. And, unfortunately for this assertion, numbers and segments of different classes *are* treated in the same discussion constantly throughout the book (see pages 28, 29, 51, 59, 77, etc.).

Finally we find, on page 196, "the limit of a (finite) variable x is a region (precisely expressed, a region of finiteness); and this region is infinitely small, but otherwise arbitrary."

Such are, indeed, in part, the necessary conclusions of a non-archimedean geometry. But it should be carefully noted that such results have nothing to do with euclidean geometry, *i. e.*, with space as *ordinarily* conceived and treated. It is the author's lack of appreciation of this fact, and his constant appeal to the intuition (and to even much less strict metaphysical persuasion) which robs the book of any value whatever, even as a discussion of non-archimedean geometry. Even the archimedean axiom itself seems to be used implicitly at

* See *c. g.*, p. 10.

points, in assuming that the geometry is continuous in the ordinary sense, but the arguments are so intermingled with intuitional conclusions that it is wholly impossible to decipher just what axioms the author is implicitly assuming. It is scarcely necessary to state that the discussions of limits, infinite series, the parallel axiom,* derivatives, etc., are riddled with errors; and the minor matters treated are no less free from gross mistakes. That Leibniz's ideas of infinity are the nearest approach to the ideas of this book, as the author frequently asserts, is in itself a sufficient condemnation.

The moral of the book concerns the thoughtless introduction of "differentials" in our ordinary books on elementary calculus.† A "differential," treated as an "infinitesimal," or "infinitely small quantity," is precisely a non-archimedean number; and no reasoning can be applied to a system of numbers in which such "infinitesimals" occur without an investigation of the extraordinary properties of such a non-archimedean (non-euclidean) arithmetic (or geometry).

On the other hand, if differentials are to be introduced in the calculus, we may do so (in two dimensions) by considering the perfectly finite quantities

$$dx \equiv \Delta x, \quad dy \equiv \frac{dy}{dx} \cdot \Delta x, \quad d^2y \equiv \frac{d^2y}{dx^2} \cdot \overline{\Delta x^2},$$

etc., where Δx is a certain fixed finite increment of x chosen at pleasure. Such a treatment can be made perfectly rigorous, and the advantages (if any such exist) of the differential notation can be preserved, while our geometry and arithmetic remain euclidean and archimedean. But the differentials here considered are perfectly finite quantities, with no taint of peculiarity or obscurity.

The errors in conceptions and in modes of thought, induced in a student by the use of differentials in a loose sense, are well illustrated by the nature of the conceptions of this author. And on serious consideration the treatment of calculus by the use of "infinitely small" differentials must be discarded ‡ by

* A "proof" of this axiom is given on p. 26. The author states that previous proofs are lacking in rigor!

† And to a minor extent, the ordinary calculus treatment of indeterminate forms, and of infinite series.

‡ This statement of course presupposes that the elementary course on the calculus is to be based on euclidean geometry, and is to use intuitive proofs of theorems involving continuity and limits, to some extent.

